A10
The Role of Transform Functions in Geophysical Inversion
T. Guenther* (Leibniz Institute for Applied Geosciences(GGA)), M. Müller-Petke (TU Berlin), M. Hertrich (ETH Zürich) & C. Rücker (University of Leipzig)

SUMMARY
We give an overview on possible transform functions (TF) used in geophysical inversion. For three different geophysical methods - dc resistivity sounding, surface nmr sounding and travel time tomography - we investigate how TF’s affect the results of synthetic data, the singular value spectrum and resolution properties.
Transform functions

In most geophysical methods we measure a quantity $r$ and try to find out about the distribution of the subsurface parameter $p$ by means of inversion methods. $p$ and $r$ are linked by formulae or differential equations. For example, in resistivity methods $p$ is the conductivity $\sigma$ and $r$ is the voltage $u$. However, in inversion we often do not use these quantities as model and data. Instead we transform them by the use of a transform function (TF). For the resistivity example the model parameter $m$ is often the resistivity $\rho = 1/\sigma$, on data side we transform the voltages to apparent resistivities $d = \rho_a = Gu/I$ by means of a geometric factor because we better understand $\rho_a$ instead of $u$. Moreover, often logarithms are chosen in order to ensure positivity. In the following, we try to give an overview on common and possible TF’s and how they affect different methods.

Basic transforms

There are a lot of simple functions that we might use for any reason. Table 1 gives an overview on basic function that might be used.

<table>
<thead>
<tr>
<th>Name</th>
<th>$m(p)$</th>
<th>$p(m)$</th>
<th>$\frac{\partial m}{\partial p}$</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>TransLinear</td>
<td>$g(p - p_0)$</td>
<td>$\frac{m}{g} + p_0$</td>
<td>$g$</td>
<td>dc resistivity $\rho_a(\frac{U}{I})$</td>
</tr>
<tr>
<td>TransPower</td>
<td>$p^n$</td>
<td>$m^{1/n}$</td>
<td>$np^{n-1}$</td>
<td>GPR: $v = \frac{1}{\sigma}$, EM: $\rho = \frac{1}{\sigma}$</td>
</tr>
<tr>
<td>TransExp</td>
<td>$m_0 e^{-p/p_0}$</td>
<td>$-p_0 \log\left(\frac{m}{m_0}\right)$</td>
<td>$-\frac{m_0}{p_0} e^{-p/p_0}$</td>
<td>amplitude</td>
</tr>
<tr>
<td>TransLog</td>
<td>$\log\left(\frac{p}{p_0}\right)$</td>
<td>$e^m$</td>
<td>$\frac{1}{p}$</td>
<td>positivity</td>
</tr>
</tbody>
</table>

Table 1: Several basic TF, their reverse functions, derivatives and examples.

Range constraints

The choice of the logarithm $m = \log(p)$ is very popular since it ensures positivity of the parameter. A lower bound $p_l$ different from zero can be applied by $m = \log(p - p_l)$. Similarly, $m = \log(p_a - p)$ defines an upper bound. Both can be combined using $m = \log(p - p_l) - \log(p_a - p) = \log \frac{p - p_l}{p_a - p}$ (logLU).

A two-side range constraint can also be implemented with trigonometric functions, e.g. $m = -\cot\left(\frac{p - p_l}{p_a - p_0} \pi\right)$ (cotLU). Both functions are (s. Fig. 1) very similar and linear in the interior and tend to plus/minus infinity at the bounds, logLU suits more for log-distributed data.

![Figure 1: Log based and cotangens based transformation functions for linearly (left) and logarithmically spaced (right) data.](image1)

Nested functions

Often we want to combine different functions, e.g. the logarithm of the (apparent) resistivity $d = \log(Gu/I)$ and $m = \log(1/\sigma)$. Also we might combine basic functions by addition or multiplication. In this case the usual rules hold. The inverse transform has then to
be done by Newton iteration generally. By the use of a generic template programming approach 
we are able to combine any two TF using the chain rule.

**Apparent parameters** A very popular data transformation is the apparent parameter approach: 
The readings are transformed into parameter space \( d = p^a \) in such a way that a homogeneous 
half-space of \( p^a \) is producing the same value. This is being used in dc resistivity or em meth-
ods, but can also be applied to any travel-time data in cross-hole tomography or even refraction 
seismic data. As a consequence, the regularisation parameter becomes unit-less and is easier to 
be chosen.

**Petrophysical relations** By petrophysical relations we derive geologic meaningful quantities. 
For example, Archie’s equation relates the electric conductivity to the water content and is based 
on transPower.

In GPR tomography obtain the water saturation \( S_w \) from the slowness \( s \). The complex refractive 
index model (CRIM) formula reads

\[
S_w = \frac{s/s_0 + \sqrt{\epsilon_m} (\Phi - 1) + \Phi}{(\sqrt{\epsilon_w} - 1)\Phi} = as + b
\]

and can be implemented by a linear transformation whose parameters depend on the permittivi-
ties \( \epsilon_i \) and porosity \( \Phi \).

Of course, petrophysical quantities can also be range-constrained, because we often know more 
about valid ranges of the water saturation instead of the slowness or effective permittivity. In 
the CRIM case, this is transNest(transLogLU,transLinear).

**Examples**

In order to investigate the role of transform functions we chose three very different geophysical 
methods: 1. a dc resistivity sounding with a layered model, a strongly non-linear problem, 
solved by the Marquardt algorithm, 2. surface NMR sounding with fixed layering and smooth-
ness constraints, which is a linear problem, and 3. 2d traveltime tomography using smoothness 
constrained, which is slightly non-linear.

In all three cases we use the following procedure: i) create a synthetic model, ii) make forward 
calculation, iii) noisify the synthetic data by Gaussian noise of realistic amount, and iv) invert 
the noisified data while the regularization strength is iterated such that the data are exactly fitted 
within noise (\( \chi^2 = 1 \)).

**1D DC resistivity sounding** The synthetic model is a three-layered case with 25, 80 and 
15 \( \Omega m \) and thicknesses of 3 and 5 m. We applied a Schlumberger sounding with \( AB/2=1 \) to 100 m 
and added 3% noise. The singular value spectrum (Fig. 2) is shown for different combinations 
of model and data. Starting from voltages and conductivity going to logarithmised resistivities 
and apparent resistivities we can see that the inverse problem is step by step improved (a more 
shallow decrease of the singular values).

Looking at the inversion results (Fig. 2 right) we can clearly see that the application of lower 
and upper bounds for the resistivity strongly enhances the quality of the result even though the 
bounds (10 and 100 \( \Omega m \)) are still far away from the synthetic model.

In order to assess the resolution properties, we show the diagonal entries of the formal resolution 
matrix (Table 2). The first and last layer resistivity and the first layer thickness is resolved very 
well. Going from log data/model to range-constrained resistivities we see a clear improvement 
of the layer thickness (particularly in the logLU version). However, the second layer resistivity 
is apparently resolved worse.
Figure 2: Singular value spectrum (left) and results for different TF (right).

<table>
<thead>
<tr>
<th>Method</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_3$</th>
<th>$d_1$</th>
<th>$d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lin-Log</td>
<td>0.99</td>
<td>0.51</td>
<td>0.98</td>
<td>0.94</td>
<td>0.46</td>
</tr>
<tr>
<td>Log-Log</td>
<td>0.99</td>
<td>0.51</td>
<td>0.98</td>
<td>0.94</td>
<td>0.47</td>
</tr>
<tr>
<td>LogLU-LogLU</td>
<td>0.99</td>
<td>0.14</td>
<td>0.98</td>
<td>0.95</td>
<td>0.85</td>
</tr>
<tr>
<td>CotLU-CotLU</td>
<td>0.99</td>
<td>0.27</td>
<td>0.98</td>
<td>0.94</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Table 2: Resolution matrix (main diagonals)

Alternatively we compute the a-posteriori model covariance matrix and transfer the square root of the diagonal matrix $\delta m$ into a parameter uncertainty $\delta \rho = \delta m / \partial \rho / \partial m$. The resistivity uncertainty for the second layer $\delta \rho_2$ is for linear data/model 59 $\Omega m$, linear data/log model 23 $\Omega m$, logLU data/model 17 $\Omega m$ and cotLU 19 $\Omega m$. Hence the model covariance matrix is a better indicator for the quality of transform functions and the logLU transform performs best in this case.

1D NMR sounding The method of nuclear magnetic resonance is the only one that yields directly a hydrological quantity, the water content WC). We regard a sounding of 24 pulse moments, the synthetic model comprises a three-layer model of 5, 30 and 15% WC. A noise of 10 nV is added and inverted with four TF’s. The results in Figure 3 show that in this case cotLU performs best, probably due to the more linear distribution of the WC.

2D GPR traveltime tomography By the traveltime of ground penetrating radar (GPR) waves we are able to determine the distribution of water in soil columns. The following examples employs a circle geometry with 24 positions for source and receiver. A very simple water saturation at a porosity of $\Phi = 0.4$ and $\epsilon_m = 5$ (eq. 1) is assumed to generate the synthetic data. Figure 4 shows the estimated water content distribution for linear data and log model (centre). Alternatively we use (eq. 1) to transform the slowness parameters into water content, the measured traveltimes into an apparent water content and constrain both by logLU to 10-50 % (right). The latter represents obviously a significant improvement.

Conclusions
Transformation functions are able to improve the images obtained by geophysical inversion. If meaningful quantities are to be derived by petrophysical relations we should incorporate them into the minimisation. On data side apparent parameter relations are useful. Range constraints on the original parameters, the petrophysical quantities, but also on the apparent parameters are
Figure 3: Synthetic model, inversion result and model rms difference (in %) for linear, log, logLU and cotLU transformations.

Figure 4: Synthetic model (left), results using traveltime/log slowness (centre) and logLU apparent water content/water content (right).

additional information that can improve the result significantly. The cotangens-based variant seems to fit best to linear parameter distribution, whereas the logarithm-based TF seems to be best for nearly lognormal distributed parameters. The improvement can be seen in the singular value spectrum and in the model covariance matrix, which proves more significant than the resolution matrix.